On the convergence to ergodic behaviour of quantum wavefunctions

1997 J. Phys. A: Math. Gen. 30 2963

(http://iopscience.iop.org/0305-4470/30/9/011)

J. Phys. A: Math. Gen. **30** (1997) 2963–2974. Printed in the UK PII: S0305-4470(97)78847-4

On the convergence to ergodic behaviour of quantum wavefunctions

Ph Jacquod† and J-P Amiet‡

Institut de Physique, Université de Neuchâtel, 1, Rue A L Breguet, CH-2000 Neuchâtel, Switzerland

Received 18 October 1996, in final form 20 January 1997

Abstract. We study the decrease of fluctuations of diagonal matrix elements of observables and of Husimi densities of quantum mechanical wavefunctions around their mean value upon approaching the semiclassical regime ($\hbar \to 0$). The model studied is a spin ($SU(2)$) in a classically strongly chaotic regime. We show that the fluctuations are Gaussian distributed, with a width of σ^2 decreasing as the square root of Planck's constant. This is consistent with the random matrix theory (RMT) predictions, and previous studies on these fluctuations. We further study the width of the probability distribution of $\hbar$-dependent fluctuations and compare it with the Gaussian orthogonal ensemble of RMT.

The behaviour of quantum mechanical wavefunctions in the semiclassical limit has recently attracted much interest. It is motivated by the fact that the spectrum alone cannot contain all the information on the system. Roughly, one can say that in integrable systems the eigenfunctions condense on classically invariant torii, while in chaotic ones, where such classical structures have been destroyed, they tend to spread uniformly over the whole classically allowed region. Few analytical results have been obtained, however, in chaotic regimes, the most important of which perhaps is the Shnirelman theorem [3]. One formulation of this theorem would be that in the limit $\hbar \to 0$, almost all the diagonal matrix elements of almost all quantum mechanical observables converge weakly to a constant over the classically chaotic region. A few years ago Feingold and Peres [2] and more recently, Eckhardt *et al* [1] studied the rate of this convergence for autonomous systems where the semiclassical limit is, according to the Shnirelman theorem, the microcanonical phase-space (i.e. classical) average. As they mentioned, 'almost all quantum mechanical observables' in this formulation exclude projection operators, and in general all operators are without smooth classical limits. Moreover, the 'almost all diagonal matrix elements' still leave room for scarring of eigenstates by short periodic orbits [4]. For those states, the limit can be dramatically different from the Shnirelman-predicted one. Their conclusion is that in a strongly chaotic system and for a smooth classical observable $A(p, q)$ with which a quantum operator $\hat{A}$, $A_{jk} := \langle E_j|\hat{A}|E_k\rangle$, can be associated, the fluctuations of the diagonal matrix elements

$$\langle F_j^2\rangle := \langle (A_{jj} - \{A\})^2\rangle \tag{1}$$

† E-mail address: philippe.jacquod@iph.unine.ch
‡ E-mail address: amiet@iph.unine.ch

around the semiclassical microcanonical average

$$\{A\} = \int A(p,q)\delta(E - H(p,q))\,\mathrm{d}^d p\,\mathrm{d}^d q \Big/ \int \delta(E - H(p,q))\,\mathrm{d}^d p\,\mathrm{d}^d q \tag{2}$$

have the same order of magnitude as the mean square of the off-diagonal terms $\langle |A_{jk}|^2\rangle$ and decrease proportionally to the inverse of the Heisenberg time $1/T_H \sim \hbar$ upon approaching the semiclassical limit, in agreement with the random matrix theory (RMT) predictions. Here $|E_{j,k}\rangle$ are energy eigenstates of the Hamiltonian under consideration, i.e. $H|E_{j.k}\rangle = E_{j,k}|E_{j,k}\rangle$ and $\langle \ldots \rangle$ means an average taken over neighbouring (in energy) eigenstates. Their arguments are valid provided E_j and E_k are not too distant from each other, but they do not need to be consecutive eigenvalues. Accordingly, the interval of energy over which the averages are taken may or may not overlap. They relate the proportionality coefficient to the autocorrelation function of the classical dynamical variable A, $C(t) := \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t+\tau)A(\tau)\,\mathrm{d}\tau$, i.e.

$$\langle F_j^2 \rangle = \frac{2}{T_H}\int_0^\infty \mathrm{d}t\, C(t). \tag{3}$$

In particular, almost all diagonal elements A_{jj} tend to the semiclassical microcanonical average as $\hbar \to 0$. Equation (3) states among others that *quantum fluctuations are proportional to classical correlations*. Their argument is as follows. According to Shnirelman's theorem, the diagonal matrix element

$$\langle E_j|\hat{A}(t)\hat{A}(0)|E_j\rangle \to C(t) \qquad \hbar \to 0. \tag{4}$$

On the other hand, this matrix element is

$$\begin{aligned}\langle E_j|\hat{A}(t)\hat{A}(0)|E_j\rangle &= \sum_k \exp[\mathrm{i}(E_j - E_k)t/\hbar]|A_{jk}|^2 \\ &= \sum_{k\neq j} \exp[\mathrm{i}(E_j - E_k)t/\hbar]|A_{jk}|^2 + |A_{jj}|^2.\end{aligned} \tag{5}$$

Thus we have

$$\sum_{k\neq j} \exp[\mathrm{i}(E_j - E_k)t/\hbar]|A_{jk}|^2 \to C(t) - \{A\}^2 \qquad \hbar \to 0. \tag{6}$$

Defining the Fourier transform of the autocorrelation function $S(\omega) := \int_{-\infty}^{\infty} C(t)\exp(-\mathrm{i}\omega t)\,\mathrm{d}t$ we have

$$|A_{jk}|^2 \approx S((E_j - E_k)/\hbar)/(2\pi\rho(E)) = \int_{-\infty}^{\infty} [C(t) - \{A\}^2]\,\mathrm{d}t \qquad E_j \to E_k. \tag{7}$$

Then, under the assumption that as $E_j \to E_k$, the eigenfunctions $|E_j\rangle$, $|E_k\rangle$ and $|\pm\rangle := \frac{1}{\sqrt{2}}(|E_j\rangle \pm |E_k\rangle)$ are qualitatively similar, i.e.

$$\begin{aligned} A_{jk} &\approx \langle -|\hat{A}|+\rangle \\ \{A\} &\approx \langle +|\hat{A}|+\rangle \approx \langle -|\hat{A}|-\rangle \end{aligned} \tag{8}$$

we have

$$A_{jk} \approx \langle -|\hat{A}|+\rangle = \tfrac{1}{2}(A_{jj} - A_{kk} + A_{jk} - A_{kj}). \tag{9}$$

Finally, by defining the fluctuations as $F_j := A_{jj} - \{A\}$ and assuming statistical independence of the F_j's, i.e. the average $\langle F_j^2\rangle = \langle F_k^2\rangle = 2\langle |A_{jk}|^2\rangle$ does not depend on the indices j and k, we get equation (3). Illustrations of this result on the double rotator model [2], the bakers map and the hydrogen atom in a magnetic field [1] nicely confirm these predictions.

These are, to our knowledge, the only works that deal with the qualitative description of the approach to ergodicity of quantum mechanical wavefunctions. Here, we extend these results to a kicked (e.g. non-autonomous) system. We will focus on the fluctuations of the Husimi density of the eigenstates, i.e. study the fluctuations of the diagonal matrix elements of the projection operator over coherent states [5]. The Hamiltonian

$$H := \frac{\hbar}{4ST} S_z^2 + \frac{\hbar\kappa}{T} S_y \sum_{n=-\infty}^{+\infty} \delta(t - nT) \qquad (10)$$

is expressed in terms of the usual $SU(2)$ spin operators S_x, S_y and S_z, while $0 \leqslant \kappa \leqslant 2\pi$. Models of this kind have been extensively studied [6] and are usually referred to as 'kicked tops'. They represent a spin which evolves during a time T under the influence of an integrable Hamiltonian after which it undergoes a rotation of the angle κ around the y-axis. It thus defines the time evolution (Floquet) operator:

$$U_T := \exp\left(-\mathrm{i}\frac{\kappa}{T} S_y\right) \exp\left(-\frac{\mathrm{i}}{4S} S_z^2\right). \qquad (11)$$

Previous investigations of this model have illustrated the remarkable agreement of its spectral properties with the GOE/COE of RMT†. In this article we will consider fluctuations of expectations values of $SU(2)$ operators taken over eigenstates of the Floquet operator (11). The above argument leading to equation (3) must be slightly modified in order to apply it to the map defined by equations (10) and (11). Instead of working with energy eigenstates $|E_j\rangle$ of an autonomous Hamiltonian, we deal with quasienergy eigenstates $|\omega_j\rangle$ of a unitary time-evolution operator. As a consequence, the microcanonical average of equation (2) is replaced by a phase-space integral restricted to the corresponding connected chaotic region. In our case and in a strongly chaotic regime equation (2) reads

$$\{A\} = \int_{\mathcal{S}^2} A(\theta, \phi) \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\phi \Big/ \int_{\mathcal{S}^2} \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\phi = \frac{1}{4\pi} \int_{\mathcal{S}^2} A(\theta, \phi) \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\phi \qquad (12)$$

i.e. we integrate over the whole sphere $\mathcal{S}^2$ instead of the energy surface. In the semiclassical limit, the diagonal matrix elements

$$\langle \omega_j | \hat{A}(t) \hat{A}(0) | \omega_j \rangle = \sum_k \exp[\mathrm{i}(\omega_j - \omega_k)t/\hbar] |A_{jk}|^2 \to C(t) \qquad (13)$$

provided the regime studied is classically strongly chaotic. Moreover, a similar argument as before leads to

$$|A_{jk}|^2 \approx S((\omega_j - \omega_k)/\hbar) \qquad (14)$$

and hence we recover equation (3). Here, we concentrate on the study of the eigenstates of the unitary operator equation (11) in the regime $T = 50$ and $\kappa = 1.2$. By standard numerical computation of the Liapounov exponent [8] over the whole phase space, we check that in this regime the classical motion is strongly chaotic. Moreover, we check that the quantum mechanical operator equation (11) exhibits the usual characteristics of quantum chaos: its level spacings statistics and spectral rigidity follow the predictions of the GOE/COE of RMT. We stress that even though the perturbation destroys the time reversibility of the system, a surviving symmetry still persists $\prod |\mu\rangle = |-\mu\rangle$. Because of the existence of this antiunitary symmetry, the model obeys GOE/COE [9].

As mentioned in [1] the Shnirelman theorem leaves room for wavefunctions to show large deviation from the semiclassical limit value. It only states that the proportion of such

† We recall the agreement of GOE (Gaussian orthogonal ensemble) and COE (circular orthogonal ensemble) properties in the limit of large matrices $N \to \infty$ [7].

wavefunctions should be negligible, i.e. in the semiclassical limit, they build a subset of zero measure. For 'almost all eigenfunctions' then, the variance of these fluctuations should vanish as $\hbar \to 0$. However, this decay can be substantially perturbed by the scarring of eigenfunctions by a short periodic orbit [4]: scarred eigenfunctions are front-line candidates for exceptions to the Shnirelman theorem! Thus, they could significantly—and negatively—affect our results. We must therefore find a way to estimate and eventually reduce the ratio of such eigenstates and to this purpose we introduce the *level curvature* [10, 11]. The level curvature is a measure of the sensitivity of an eigenvalue to an external perturbation. In our model (10) for instance we can define it as the second derivative of an eigenvalue with respect to κ

$$K_n = \frac{\mathrm{d}^2\omega_n\,(\kappa, T)}{\mathrm{d}\kappa^2}. \tag{15}$$

Intuitively, when the studied regime is highly chaotic, the spectrum shows a level repulsive behaviour which results in a number of avoided crossings when varying one parameter. In the direct vicinity of an avoided crossing, the curvature of two levels can be huge and therefore the distribution of these values depends very sensitively on the regime studied, i.e. on both κ and T.

Scarred eigenstates shift almost linearly in energy when varying one parameter and hence have generally small level curvatures. Consequently, it has been suggested that scarring manifests itself in deviations of RMT predictions in the level curvature distribution [10, 11]. Though not yet rigorously proven, this statement is now widely accepted. This distribution for the model defined by equation (11) in the regime studied is shown in figure 1. There is a remarkable agreement with the GOE/COE (full curve) prediction [10]

$$P(k) = \frac{1}{2}\frac{1}{(1+k^2)^{3/2}} \qquad k = \frac{K}{\pi\,\bar{\rho}\langle(\frac{\mathrm{d}\omega_n}{\mathrm{d}\kappa})^2\rangle}. \tag{16}$$

Here $\bar{\rho} = 2\pi/(2s+1)$ is the averaged level density. This indicates a small number of scarred eigenstates, an agreement which was already obtained on a similar model in [10]. Therefore, scarring is not likely to influence our study.

Let us briefly outline our method. Our aim is to study the behaviour of the eigenstates of equation (11) in the semiclassical limit, i.e. as $\hbar \to 0$, $S \to \infty$ so as to leave the product $\hbar S$ constant. A peculiarity of such systems is that the parameter governing the convergence to the semiclassical limit also governs the number of states $2S + 1 \sim 1/\hbar$ and the density of states. In order to determine the implication of this peculiarity on our study, we will therefore check the validity of our results on GOE matrices.

The Husimi density of an eigenstate $|\omega\rangle = \sum_{\mu=-S}^{S} \omega_\mu |\mu\rangle$ of equation (11) is defined as the projection of this state onto a coherent state $|\theta, \phi\rangle$ of the spin $SU(2)$ group [5]:

$$\begin{aligned} &\Omega_\omega^S(\theta,\phi) := |\langle\omega|\theta,\phi\rangle|^2 \\ &|\theta,\phi\rangle := \sum_{\mu=-s}^{s} \sqrt{\begin{pmatrix} 2s \\ s-\mu \end{pmatrix}} \sin\left(\frac{\theta}{2}\right)^{s-\mu} \cos\left(\frac{\theta}{2}\right)^{s+\mu} \mathrm{e}^{\mathrm{i}(s-\mu)\phi}|\mu\rangle. \end{aligned} \tag{17}$$

The Husimi density satisfies the assumptions of the Shnirelman theorem [3]. Indeed one formulation of the latter refers to the uniform spreading of eigenstates $|\Psi_{\mathrm{chaos}}\rangle$ over a connected chaotic region of phase space which implies that

$$\Omega_\Psi^S(\theta,\phi) = |\langle\Psi_{\mathrm{chaos}}|\theta,\phi\rangle|^2 \longrightarrow \begin{cases} 0 & \text{on the regular region} \\ \text{constant} & \text{on the chaotic region.} \end{cases} \tag{18}$$

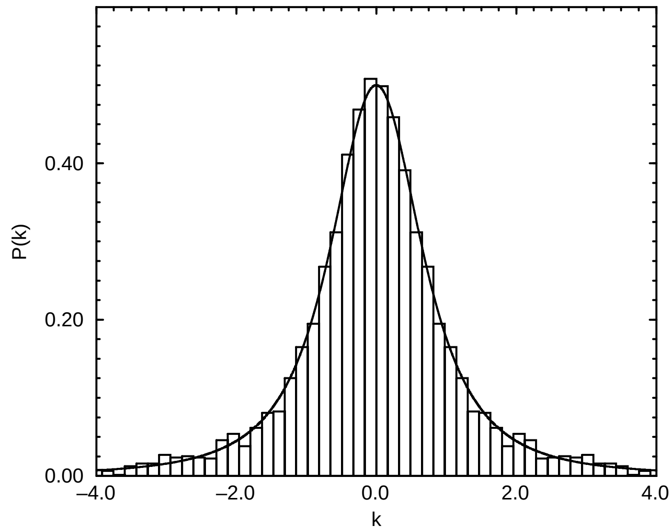

Figure 1. Distribution of level curvatures for the eigenstates of (11) and $S = 200$, $T = 50$ and $\kappa = 1.2$. From the remarkable agreement with RMT predictions we conclude that the ratio of scarred eigenfunctions is very small (see [10, 11]), and should therefore not influence our study.

Thus in our model, as $\hbar \to 0$, the Husimi density converges weakly to a constant over the whole phase space. The Ω^S_ω are smooth functions of θ and ϕ and thus can be expanded in a multipole expansion over the basis of spherical harmonics:

$$\Omega^S_\omega(\theta, \phi) = \sum_{l,m} \sqrt{\frac{4\pi}{2l+1}} \Omega^S_{l,m} Y_{l,m}(\theta, \phi) \tag{19}$$

where $l = 0, 1, 2, \ldots, 2S$ and $m = -l, -l+1, -l+2, \ldots, l$ and $|\mu\rangle$ is an eigenstate of S_z, i.e. $S_z|\mu\rangle = \mu|\mu\rangle$. We used the convention to introduce the square root into this expansion. This multipole expansion allows us to interpret the $\Omega^S_{l,m}$ in terms of magnitude of fluctuations of size $\sim \frac{\pi}{m+1}$ in the ϕ-direction and $\sim \frac{2\pi}{l+1}$ in the θ-direction. We will thus get quantitative results on the decrease of fluctuations as a function of their size. Let us recall that the Shnirelman theorem implies that as $\hbar = 1/S \to 0$, fluctuations of fixed and non-zero l must vanish, i.e. $\Omega^S_\omega(\theta, \phi) \to \Omega^S_{0,0}$. However, it does not say anything about the behaviour of, say, $\Omega^S_{l(S),m(S)}$ as $S \to \infty$ when $l(S)$ and $m(S)$ are monotonically increasing functions of S, i.e. investigating such multipoles could lead us to different conclusions than that of [1, 2].

Using the resolution of unity

$$\mathbf{1} = \frac{2s+1}{4\pi} \int \mathrm{d}\theta\, \mathrm{d}\phi\, \sin\theta |\theta, \phi\rangle\langle\theta, \phi| \tag{20}$$

the normalization condition reads

$$\begin{aligned} 1 &= \langle\omega|\mathbf{1}|\omega\rangle = (2S+1)\Omega^S_{0,0} \\ \Rightarrow \Omega^S_{0,0} &= \frac{1}{4(2S+1)} \end{aligned} \tag{21}$$

i.e. the zeroth moment decreases as $1/S \sim \hbar$ on approaching the semiclassical limit. Let us note that this $1/S$ behaviour of the Shnirelman limit $\Omega^S_{0,0}$ is a consequence of the overcompleteness of the coherent states representation which necessitates the $2s + 1$ factor in the resolution of unity (20), hence it has no direct physical meaning. In the following, we therefore divide all higher multipoles $\Omega^S_{l,m}$ by $\Omega^S_{0,0}$ to consistently study their decrease and introduce the notation $\hat{\Omega}^S_{l,m} := \frac{\Omega^S_{l,m}}{\Omega^S_{0,0}}$. On the other hand, we have,

$$\hat{\Omega}^S_{l,m} = 4(2S+1)\sum_{\mu=-s}^{s}\omega^*_\mu\omega_{\mu+m}(-1)^{s-\mu}C^{s,s,l}_{s,-s,0}C^{s,s,l}_{\mu+m,-\mu,m} \tag{22}$$

which gives a check of our numerical computation for small S. However, the numerical difficulty for computing the Clebsch–Gordan coefficients $C^{s,s,l}_{\mu+m,-\mu,m}$ for large S leads us to use the following numerically more stable and faster method to compute multipoles $\hat{\Omega}^S_{l,m}$. We define

$$M^S_{k,m}(\omega) := 4(2S+1)\langle\omega|S^k_z S^m_-|\omega\rangle/S^{k+m}. \tag{23}$$

It is straightforward to see that there is a linear relation between the $M^S_{k,m}$ and the $\hat{\Omega}^S_{l,m}$ (we use the shorter notation $\gamma = \mathrm{e}^{\mathrm{i}\phi}\tan(\theta/2)$)

$$\begin{aligned} M^S_{k,m}(\omega) &= \frac{1}{S^{k+m}}\,\mathrm{Tr}[|\omega\rangle\langle\omega|S^k_z S^m_-] \\ &= \frac{2S+1}{4\pi\, S^{k+m}}\int \mathrm{d}^2\gamma\,\langle\gamma|\omega\rangle\langle\omega|S^k_z S^m_-|\gamma\rangle \\ &= \frac{2S+1}{4\pi\, S^{k+m}}\int \mathrm{d}^2\gamma\,\hat{\Omega}^S_\omega(\gamma)\circ(\mathcal{S}_z\circ)^k(\mathcal{S}_-\circ)^m. \end{aligned} \tag{24}$$

The script letters $\mathcal{S}$ stand for classical quantities, and for any function $f(\gamma)$ we have defined the product [12]:

$$\begin{aligned} f(\gamma)\circ\mathcal{S}_z &:= \left(\mathcal{S}_z - \gamma\frac{\partial}{\partial\gamma}\right)f(\gamma) \\ f(\gamma)\circ\mathcal{S}_- &:= \left(\mathcal{S}_- + \frac{\partial}{\partial\gamma}\right)f(\gamma). \end{aligned}$$

This allows us to write $\langle\gamma|\omega\rangle\langle\omega|S^k_z S^m_-|\gamma\rangle$ as a differential operator acting on $\hat{\Omega}^s_\omega(\gamma)$. The trick is then to partially integrate this expression. After a little algebra we reach

$$\begin{aligned} &\frac{2S+1}{4\pi\, S^{k+m}}\int \mathrm{d}^2\gamma\,\hat{\Omega}^S_\omega(\gamma)\circ(\mathcal{S}_z\circ)^k(\mathcal{S}_-\circ)^m \\ &\quad= \frac{(2S+1+m)!}{(2S)!2^{k+m}4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}u\,\mathrm{e}^{-\mathrm{i}m\phi}(1-u^2)^{m/2}\mathcal{P}^S_{k,m}(u)\hat{\Omega}^S_\omega \end{aligned} \tag{25}$$

where

$$\mathcal{P}^S_{k,m}(u) = \left((2S+2+m)u - m - (1-u^2)\frac{\mathrm{d}}{\mathrm{d}u}\right)^k 1 := \sum_{l'=m}^{k+m} p^{l'}_k P^m_{l'}(u) \tag{26}$$

in terms of the Legendre polynomials $P^m_{l'}(u)$ and $u = \cos(\theta)$. We finally get

$$M^S_{k,m}(\omega) = \frac{(2S+1+m)!}{(2S)!(2S)^{k+m}}\sum_{l=m}^{k+m}\frac{1}{2l+1}\hat{\Omega}^S_{l,m}p^l_k. \tag{27}$$

It is thus possible to obtain the $\hat{\Omega}^S_{l,m}$ through a matrix multiplication of the moments $M^S_{k,m}(\omega)$

$$M^S(\omega) = \mathcal{M}\hat{\Omega}^S \tag{28}$$

where we defined $(M^S(\omega))_{k,m} = M^S_{k,m}(\omega)$, $(\hat{\Omega}^S)_{l,m} = \frac{(2S+1+m)!}{(2S)^m}\hat{\Omega}^S_{l,m}$ and $(\mathcal{M})_{k,l} = \frac{1}{(2S)!(2S)^k}\frac{1}{2l+1}p^l_k$.

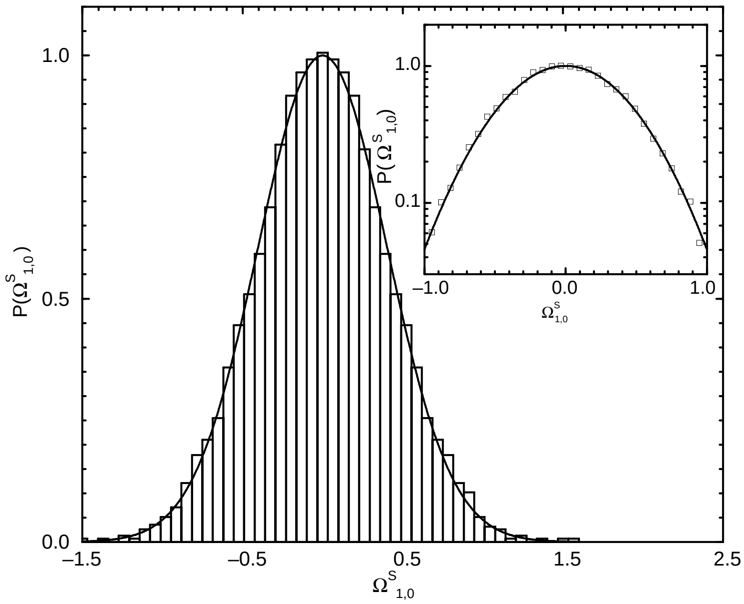

Figure 2. Moment distribution $P(\hat{\Omega}^S_{1,0})$ as defined in (16) for a spin $S = 600$. The statistics has been computed from 2404 even states of four realizations of (11) taken around $T = 50$ and $\kappa = 1.2$. The agreement with a Gaussian (full curve) is remarkable. In the inset we show the same curve on a semi-log plot.

Numerical inversion of this last matrix allows us to get the multipoles $\hat{\Omega}^S_{l,m}$ from the numerical computation of the moments $M^S_{k,m}(\omega)$. The advantage of this method compared with the direct computation of Husimi densities is the numerical stability. Moreover, if we are interested in the first few multipoles, say up to $l \ll S$, then only the diagonal matrix elements up to $M^S_{l,m}$ are necessary.

Figure 2 shows a plot of a moment distribution $P(\hat{\Omega}^{600}_{1,0})$ obtained through computation of 2404 diagonal matrix elements from four unitary matrices defined by equation (11)† close to the regime $T = 50$ and $\kappa = 1.2$. The agreement with the Gaussian fitting is remarkable and allows us to conjecture that the fluctuations of the $\hat{\Omega}^S_{l,m}$ obey the probability distribution

$$P(\hat{\Omega}^S_{l,m}) \propto \exp(-(\hat{\Omega}^S_{l,m} - \hat{\Omega}^\infty_{l,m})^2/(2\sigma^2_{l,m})) \tag{29}$$

where the mean value $\hat{\Omega}^\infty_{l,m}$ is the Shnirelman limit. This distribution narrows itself as $\hbar \to 0$, until finally the 'almost all' wavefunctions, i.e. those which obey the Shnirelman theorem, have converged to their Shnirelman limit $\hat{\Omega}^\infty_{l,m} = 0$, $l \neq 0$ and $\hat{\Omega}^\infty_{0,0} = 1$. In other words, $\sigma^2_{l,m}$ decreases as S increases. This decay follows a power law as shown in figure 3. We have

$$\sigma^2_{l,m} \sim S^{-1/2} \qquad \forall l \neq 0. \tag{30}$$

As already mentioned, this law is valid for fixed l and m in the regime $l, m \ll S$.

We further did the same study on GOE matrices. We constructed the M-matrix defined in equation (23) using eigenstates of a GOE matrix instead of the eigenstates $|\omega\rangle$ of the kicked top (11). The result is shown in figure 4 and indicates a decay of the width of the

† We have considered only the projection of (11) on even states, i.e. states which are left invariant by the parity $\mathbf{\Pi}|\mu\rangle = |-\mu\rangle$.

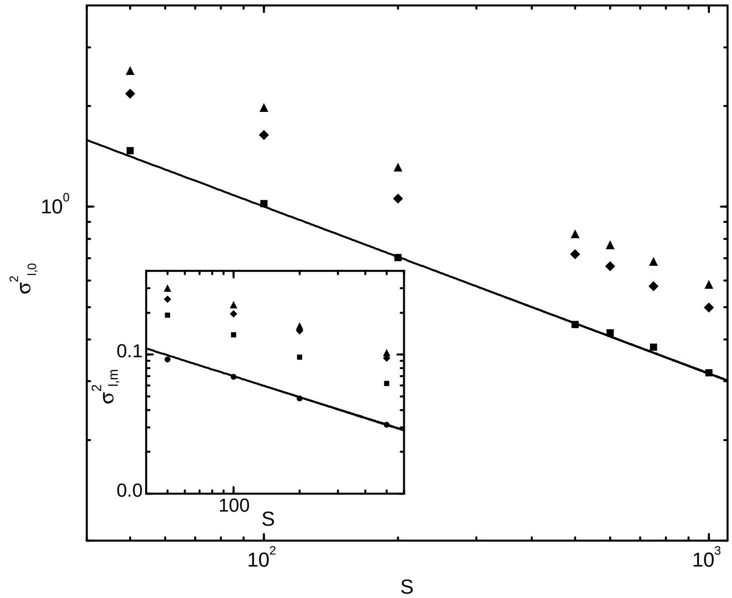

Figure 3. Log–log plot of the width of the Gaussian distribution of multipoles $P(\hat{\Omega}^S_{l,m})$ for model (11), $m = 0$ and $l = 1$ (squares), $l = 3$ (diamonds) and $l = 5$ (triangles) versus the magnitude of spin S. The inset shows the width of $P(\mathrm{Re}(\Omega^S_{l,m}))$ for $m = 2$ and $l = 2$ (circles), $l = 3$ (squares), $l = 4$ (diamonds) and $l = 5$ (triangles). In both cases, the full curve indicates the $S^{-1/2}$ decay.

Gaussian distribution of fluctuations of the Husimi density of the form (29). Let us note at this stage that the relationship between this width and the fluctuations of observables similar to those studied in [1, 2], is

$$(\sigma^2_{l,m})^2 \sim \langle F_j^2 \rangle. \tag{31}$$

Indeed $\sigma^2_{l,m}$ measures the fluctuations of the Husimi density. They are linearly related to the matrix elements of observables according to equations (23) and (27). The fluctuations of these matrix elements are roughly given by their square and hence we get equation (31). We thus get the same $1/S$ decay of the fluctuations as in [1, 2]. In other words, *the Husimi density converges to its semiclassical value with a rate given by the square root of the rate of convergence of diagonal matrix elements of observables.* This rate is independent of the size of the fluctuations. As for the shape of these fluctuations, the diversity of models studied to date leads us to conjecture that *quantum mechanical systems with strongly chaotic classical counterparts have Gaussian-distributed fluctuations of their diagonal matrix elements around their microcanonical classical average (equation (2) or (12)).* Apparently, the width of this Gaussian decays like $\hbar$ as $\hbar \to 0$. This postulate is to be taken with the 'almost all' Shnirelman restrictions and excludes of course models such as the kicked rotator [14], where quantum interference effects lead to localization of the wavefunction, thus destroying the ergodicity of the quantum wavefunction†. In the classically strongly chaotic regime we are dealing with here, the 'localization length' in the kicked top exceeds by far the total number of eigenstates $2s + 1$, hence no localization effect occurs [13].

† However, restriction of quantum averages to phase-space region smaller than the localization length should lead to a similar behaviour.

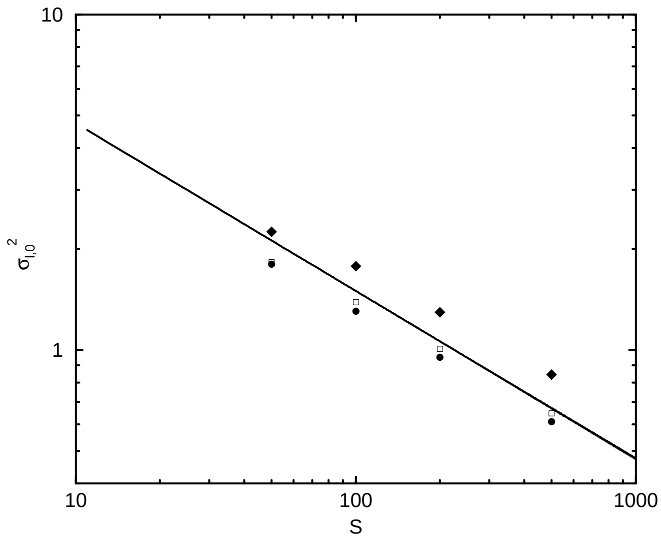

Figure 4. Log–log plot of the width of the Gaussian distribution of multipoles $P(\hat{\Omega}^S_{l,m})$ for GOE, $m = 0$ and $l = 1$ (circles), $l = 3$ (squares) and $l = 5$ (diamonds) versus the magnitude of spin S. The full line indicates the $S^{-1/2}$ decay.

Up to now we have shown that our model matches in every respect all the features of a GOE random matrix: its spectrum exhibits level repulsion, its level curvature statistics correspond to the RMT predicted distribution, and the statistical distribution of the components of its eigenvectors tends to the semiclassical average in the same way, which in its turn implies a decay of the width of the Gaussian distribution of the multipoles $\hat{\Omega}^S_{l,m}$ defined in (19). However, as has already been said, there is absolutely no reason to expect a similar decrease when l is not small compared with S. We therefore turn our attention to the behaviour of these multipoles.

We concentrate on the questions:

• is there a similar power-law decay for $\hat{\Omega}^S_{l(S),m(S)}$ when $l(S)$ and $m(S)$ are increasing functions of S?

• Are there possibly restrictions on $l(S)$ and $m(S)$ for this power law to remain valid?

Answering these questions gives us information on the minimal size $\Delta_{l,m}$ of the relevant fluctuations. From the Heisenberg uncertainty principle, quantum mechanics does not resolve details smaller than $\hbar^d$ in the $2d$-dimensional phase space of a d-dimensional system. Hence, we have a lower bound for the fluctuations size $\Delta_{l,m} = \frac{2\pi^2}{(l+1)(m+1)} \leqslant \hbar \sim 1/S$ and thus an upper bound for l and m: $l, m \leqslant S$. For the sake of simplicity we will restrict ourselves to the study of $m = 0$ multipoles with $l \sim S$ and $\sqrt{S}$ using formula (22) with random eigenfunction components ω_μ which corresponds to the GOE case†.

We show the result of this study in figure 5 for $l(S) = S/2$, $3S/4$, S and $5S/4$. Obviously, these S-dependent multipoles decay faster than those with fixed l and m. Moreover, a S_c is likely to exist for each $l(S)$ above which the magnitude of the corresponding fluctuation decays faster than a power law, possibly exponentially. However,

† The ω_μ's are random up to the normalization condition $\sum_{\mu=-S}^{S} |\omega_\mu|^2 = 1$ and the Π-parity: $\omega_\mu \neq 0$ either for $\mu = -S, -S+2, -S+4, \ldots, S$ or $\mu = -S+1, -S+3, \ldots, S-1$.

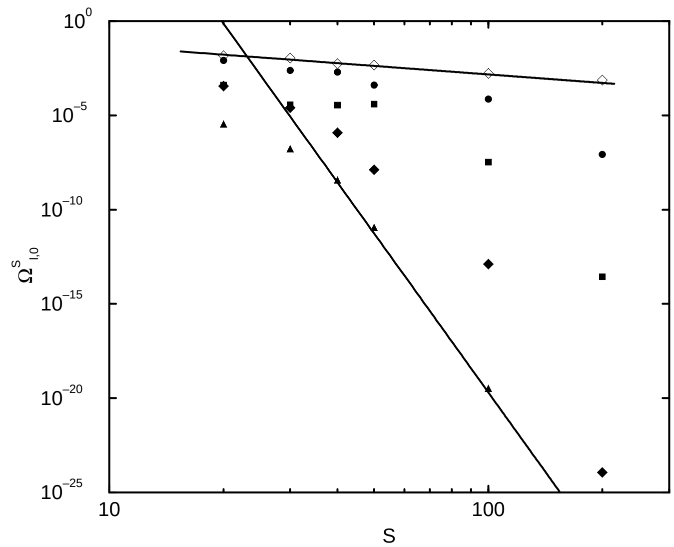

Figure 5. Log–log plot of the width of the Gaussian distribution of multipoles $P(\hat{\Omega}^S_{l,m})$ for GOE, $m = 0$ and $l = S/2$ (circles), $l = 3S/4$ (squares), $l = S$ (diamonds), $l = 5S/4$ (triangles) and $l = \sqrt{S}$ (open diamonds) versus the magnitude of spin S. The upper and lower full lines indicate a decay of $S^{-1.5}$ and S^{-36} respectively.

this latter conclusion must be taken carefully because of the restricted S-range of figure 5†. On the other hand, the $l = \sqrt{S}$-moment decay as a power law $\sim S^{-3/2}$, at least in the studied range of variation of S.

In view of this, we conclude that, in the GOE case, the critical value l_c below which the fluctuations are relevant either tends to a constant, or to infinity slower than S, i.e.

$$l_c \sim S^{\alpha} \qquad 0 < \alpha < 1. \tag{32}$$

On the other hand, a previous study [15] of the kicked top emphasized the quasifractal structure of the Husimi density of its eigenfunctions in the chaotic regime. This means that fluctuations in both directions of phase space are present up to the smallest scale allowed by the Heisenberg uncertainty, i.e. up to a size $O(\hbar^{1/2})$, which is consistent with equation (32) with $\alpha = 0.5$. The fact that the moment $\Omega_{\sqrt{S},0}$ also shown in figure 5 decays more or less as a power law

$$\Omega_{\sqrt{S},0} \sim S^{-3/2} \tag{33}$$

corroborates this conclusion: multipoles up to $l \sim \sqrt{S}$ are relevant, i.e. $\alpha = \frac{1}{2}$.

Nevertheless, nothing forces the eigenstates of a quantum chaotical model to match those of a GOE matrix up to the smallest scales. It would therefore be highly desirable to get a condition on α like equations (32), (33) for a quantum chaotical system. This could be achieved by direct computation of $\Omega_{\sqrt{S},m(S)}$ by using equation (22). However, the numerical difficulty associated with the computation of high-order Clebsch–Gordan coefficients renders this task hardly fulfillable, as can be seen in figure 6 where we show results obtained for $\Omega_{\sqrt{S},0}$ through equation (22) averaged over more than 40 000 states for each point. On one hand, the semiclassical randomness of the eigenstates is not attained for small S, while on the other hand, the Clebsch–Gordan coefficients limit the maximal spin magnitude. In other

† This restriction is due to the computation of the Clebsch–Gordan coefficients.

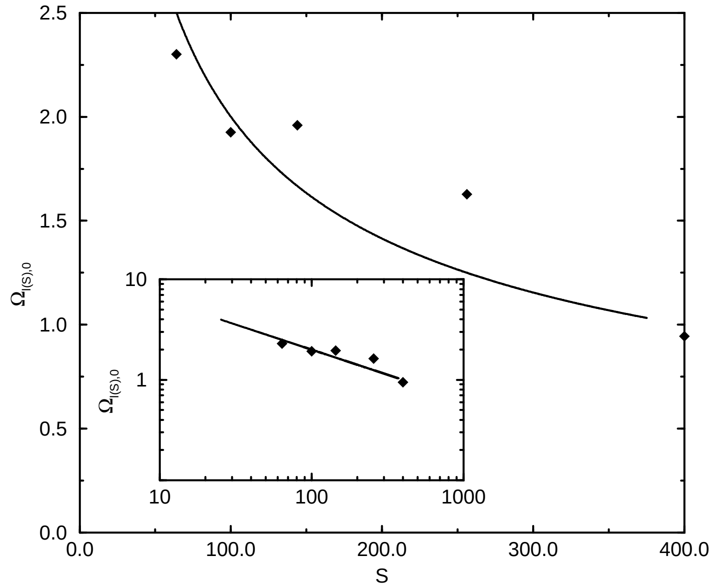

Figure 6. Log–log plot of the width of the Gaussian distribution of multipoles $P(\hat{\Omega}^S_{l(S),0})$ for model (11), $l(S) = \sqrt{S}$ and $T = 50$ and $\kappa = 1.2$, $m = 0$ and $S = \sqrt{S}$ versus the magnitude of spin S. The full curve indicates a decay of $S^{-1/2}$. We attribute the rather erratic behaviour of the data to the numerically instable computation of high-order Clebsch–Gordan coefficients (see text).

words, these two effects dramatically affect figure 6 left and right. Considering the size of our statistics, we attribute to these effects the somehow erratic behaviour of $\Omega_{\sqrt{S},0}$. In figure 6, the full line indicating a $S - 1.5$ decay is shown as an eye guide, and constitutes in no way a serious result.

In conclusion, our study of the Husimi density of eigenstates of the quantum spin system defined by (10) and (11) has confirmed the Gaussian shape of fluctuations around the semiclassical limit. These fluctuations decay in size with a rate of $\sim 1/\sqrt{S}$ for $l \ll l_c \sim \sqrt{S}$. This rate possibly increases to $1/S^{3/2}$ when l is smaller but of the order of l_c. Moreover, this decay results in the same power law for the decay of fluctuations of diagonal matrix elements of observables as in previous studies [1, 2], indicating perhaps universality. While GOE results tend to confirm the quasifractality proposed in [15], numerical difficulties forbade us to check it for the quantum dynamical system. Investigations to overcome this difficulty are in progress. For the time being, let us just point out that the fact that GOE eigenstates which seem to exhibit this quasifractality render it as a direct consequence of the randomness of the states. The maximal randomness is then bounded by Heisenberg's uncertainty, but beside that, the quasifractality of the states seems to have no physical content.

Acknowledgments

We thank the Centro Svizzero di Calcolo Scientifico. This work was supported in part by the Fonds National Suisse de la Recherche Scientifique.

References

[1] Eckhardt B, Fishman S, Keating J, Agam O, Main J and Müller K 1995 *Phys. Rev.* E **52** 5893
[2] Feingold M and Peres A 1986 *Phys. Rev.* A **34** 591
[3] See the addendum of Shnirelman A I 1993 *KAM Theory and Semiclassical Approximations to Eigenfunctions, V.F Lazutkin* (Berlin: Springer)
[4] Heller E J 1984 *Phys. Rev. Lett.* **53** 1515
[5] Perelomov A 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
[6] Jacquod Ph and Amiet J-P 1995 *J. Phys. A: Math. Gen.* **28** 4799 and references therein
[7] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[8] See e.g. Lichtenberg A J and Lieberman M A 1992 *Regular and Chaotic Dynamics* 2nd edn (Berlin: Springer)
[9] Robnik M and Berry M V 1986 *J. Phys. A: Math. Gen.* **19** 669
[10] Zakrzewski J and Delande D 1993 *Phys. Rev.* E **47** 1650 and references therein
[11] Takami T and Hasegawa H 1992 *Phys. Rev. Lett.* **68** 419
[12] Amiet J-P and Cibils M 1991 *J. Phys. A: Math. Gen.* **24** 1515
[13] For a comparison of the kicked rotator and the kicked top, see e.g. Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
[14] Shepelyansky D L 1987 *Physica* D **28** 103
[15] Nakamura K, Okazaki Y and Bishop A R 1986 *Phys. Rev. Lett.* **57** 5